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Einstein-Bianchi Hyperbolic System for General Relativity[†]

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Abstract

By employing the Bianchi identities for the Riemann tensor in conjunction with the Einstein equations, we construct a first order symmetric hyperbolic system for the evolution part of the Cauchy problem of general relativity. In this system, the metric evolves at zero speed with respect to observers at rest in a foliation of spacetime by spacelike hypersurfaces while the curvature and connection propagate at the speed of light. The system has no unphysical characteristics, and matter sources can be included.

1 Introduction

We consider the evolution part of the Cauchy problem in General Relativity[1] as the time history of the two fundamental forms of a spacelike hypersurface: its metric $\bar{\mathbf{g}}$ and its extrinsic curvature $\mathbf{K}$. On such a hypersurface, for example an "initial" one, these two quadratic forms must satisfy four initial value or constraint equations. These constraints can be posed and solved as an elliptic system by known methods that will not be discussed here. (See, for example, [1].)

The Ricci tensor of the spacetime metric can be displayed in a straightforward 3+1 decomposition giving the time derivatives of $\bar{\mathbf{g}}$ and $\mathbf{K}$ in terms of the space derivatives of these quantities. These expressions contain also the lapse and shift functions characterizing the threading of the spacelike hypersurfaces by time lines. However, proof of the existence of a causal evolution in local Sobolev spaces into an Einsteinian spacetime does not result directly from these equations, which do not form a hyperbolic system for arbitrary lapse and shift, despite the fact that their characteristics are only the light cone and the normal to the time slices[2].

In this paper we present in detail, using "algebraic gauge" or a generalized harmonic time slicing condition on the lapse function, a hyperbolic system for these geometric unknowns that is based directly on the Riemann tensor, using the Bel[3] electric and magnetic fields and inductions, and the full Bianchi identities. Its characteristics are the physical light cone and the time direction orthogonal to the spacelike submanifolds M_t of the chosen slicing. The spacetime curvature tensor is among the quantities that are evolved explicitly. Its propagation is coupled to that of the connection, extrinsic curvature, and metric of M_t through relations introduced by H. Friedrich[2], who obtained an analogous vacuum system based on the Weyl tensor, causal but with additional characteristics. Our system constitutes a complete first order symmetric hyperbolic system equivalent to the usual Einstein equations including matter. This new system, which was sketched briefly in [4], is completely equivalent to our previous one [5, 6, 7]. The space coordinates and the shift three-vector are arbitrary; and, in this sense, the system is independent of a gauge choice. In fact, we can also make the lapse function take arbitrary values by introducing a given arbitrary function into its definition. Indeed, our procedure for the lapse function is equivalent to specifying $\bar{g}^{-1/2}N$ as an arbitrary positive function of the spacetime coordinates.

2 Metric, Connection, and Curvature

It is convenient to write the metric on the hyperbolic spacetime manifold $M_t \times R$ as

$$ds^2 = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j, \tag{1}$$

with $\theta^0 = dt$ and $\theta^i = dx^i + \beta^i dt$, where β^i is the shift vector. The cobasis θ^α satisfies

$$d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta \wedge \theta^\gamma, \tag{2}$$

with $C^i{}_{0j} = -C^i{}_{j0} = \partial_j \beta^i$ and all other structure coefficients zero. The corresponding vector basis is given by $e_0 = \partial_t - \beta^j\partial_j$, where $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x^i$, and the action of e_0 on space scalars is the Pfaffian or convective derivative

$$e_0[f] = \partial_0 f = \partial_t f - \beta^j \partial_j f.$$

We shall assume throughout that the lapse function $N > 0$ and the space metric $\bar{\mathbf{g}}$ on M_t is properly Riemannian. Note that $\bar{g}_{ij} = g_{ij}$ and $\bar{g}^{ij} = g^{ij}$. (An overbar denotes a spatial tensor or operator.)

The connection one-forms are defined by

$$\omega^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} + g^{\alpha\delta}C^\epsilon{}_{\delta(\beta}g_{\gamma)\epsilon} - \frac{1}{2}C^\alpha{}_{\beta\gamma}, \tag{3}$$

where $(\beta\gamma) = \frac{1}{2}(\beta\gamma + \gamma\beta)$, $[\beta\gamma] = \frac{1}{2}(\beta\gamma - \gamma\beta)$, and Γ denotes a Christoffel symbol. We adhere to the convention

$$\nabla_\alpha v^\beta = \partial_\alpha v^\beta + v^\sigma \omega^\beta{}_{\sigma\alpha} \tag{4}$$

for the spacetime covariant derivative ∇_α and likewise for the spatial covariant derivative $\bar{\nabla}_i$. The Riemann curvature tensor is given by

$$R^\alpha{}_{\beta,\rho\sigma} = \partial_\rho\omega^\alpha{}_{\beta\sigma} - \partial_\sigma\omega^\alpha{}_{\beta\rho} + \omega^\alpha{}_{\lambda\rho}\omega^\lambda{}_{\beta\sigma} - \omega^\alpha{}_{\lambda\sigma}\omega^\lambda{}_{\beta\rho} - \omega^\alpha{}_{\beta\lambda}C^\lambda{}_{\rho\sigma} \tag{5}$$

and the corresponding Ricci identity is

$$\nabla_\alpha\nabla_\beta u_\gamma - \nabla_\beta\nabla_\alpha u_\gamma = R_{\alpha\beta,\gamma}{}^\delta u_\delta. \tag{6}$$

(The comma in $R_{\alpha\beta,\gamma\delta}$ is used to distinguish the two antisymmetric index pairs $[\alpha\beta]$ and $[\gamma\delta]$.)

The connection coefficients are written in 3+1 form as

$$\begin{aligned}
\omega^i{}_{jk} &= \Gamma^i{}_{jk} = \bar{\Gamma}^i{}_{jk}, \\
\omega^i{}_{j0} &= -NK^i{}_j + \partial_j \beta^i, \\
\omega^i{}_{0j} &= -NK^i{}_j, \\
\omega^i{}_{00} &= Ng^{ij}\partial_j N, \\
\omega^0{}_{0i} &= \omega^0{}_{i0} = N^{-1}\partial_i N, \\
\omega^0{}_{00} &= N^{-1}\partial_0 N,
\end{aligned} \tag{7}$$

and

$$\omega^0{}_{ij} = \frac{1}{2}N^{-2}\hat{\partial}_0 g_{ij} = -N^{-1}K_{ij}, \tag{8}$$

where K_{ij} is the extrinsic curvature (second fundamental tensor) of M_t and for any t-dependent space tensor $\mathbf{T}$, we define another such tensor $\hat{\partial}_0\mathbf{T}$ of the same type by setting, as in (8),

$$\hat{\partial}_0 = \frac{\partial}{\partial t} - \mathcal{L}_\beta, \tag{9}$$

where $\mathcal{L}_\beta$ is the Lie derivative on M_t with respect to the spatial vector β. Note that $\hat{\partial}_0$ and $\partial_i = \partial/\partial x^i$ commute.

3 Bianchi Equations

We recall that the Riemann tensor of a pseudo-Riemannian metric satisfies the Bianchi identities

$$\nabla_\alpha R_{\beta\gamma,\lambda\mu} + \nabla_\gamma R_{\alpha\beta,\lambda\mu} + \nabla_\beta R_{\gamma\alpha,\lambda\mu} \equiv 0. \tag{10}$$

These identities imply by contraction and use of the symmetries of the Riemann tensor

$$\nabla_\alpha R^\alpha{}_{\beta,\lambda\mu} + \nabla_\mu R_{\lambda\beta} - \nabla_\lambda R_{\mu\beta} \equiv 0, \tag{11}$$

where the Ricci tensor is defined by

$$R^\alpha{}_{\beta,\alpha\mu} = R_{\beta\mu}.$$

If the Ricci tensor satisfies the Einstein equations

$$R_{\alpha\beta} = \rho_{\alpha\beta}, \tag{12}$$

then the previous identities imply the equations

$$\nabla_\alpha R^\alpha{}_{\beta,\lambda\mu} = \nabla_\lambda \rho_{\mu\beta} - \nabla_\mu \rho_{\lambda\beta}. \tag{13}$$

Equation (10) with $\{\alpha\beta\gamma\} = \{ijk\}$ and (13) with $\beta = 0$ do not contain derivatives of the Riemann tensor transversal to M_t; hence, we consider these equations as constraints ("Bianchi constraints"). We shall consider the remaining equations in (10) and (13) as applying to a double two-form $A_{\alpha\beta,\lambda\mu}$, which is simply a spacetime tensor antisymmetric in its first and last pairs of indices. We do *not* suppose *a priori* a symmetry between the two pairs of antisymmetric indices. These equations, called from here on "Bianchi equations," read as follows

$$\nabla_0 A_{hk,\lambda\mu} + \nabla_k A_{0h,\lambda\mu} + \nabla_h A_{k0,\lambda\mu} = 0, \tag{14}$$

$$\nabla_0 A^0{}_{i,\lambda\mu} + \nabla_h A^h{}_{i,\lambda\mu} = \nabla_\lambda \rho_{\mu i} - \nabla_\mu \rho_{\lambda i} \equiv J_{\lambda\mu i}, \tag{15}$$

where the pair $[\lambda\mu]$ is either $[0j]$ or $[jl]$. We next introduce[3] two "electric" and two "magnetic" space tensors associated with the double two-form $\mathbf{A}$, in analogy to the electric and magnetic vectors associated with the electromagnetic two-form $\mathbf{F}$. That is, we define the "electric" tensors by

$$E_{ij} \equiv A^0{}_{i,0j} = -N^{-2} A_{0i,0j}, \tag{16}$$

$$D_{ij} \equiv \frac{1}{4} \eta_{ihk} \eta_{jlm} A^{hk,lm},$$

while the "magnetic" tensors are given by

$$H_{ij} \equiv \frac{1}{2} N^{-1} \eta_{ihk} A^{hk}{}_{,0j}, \tag{17}$$

$$B_{ji} \equiv \frac{1}{2} N^{-1} \eta_{ihk} A_{0j,}{}^{hk}.$$

In these formulae, η_{ijk} is the volume form of the space metric $\bar{\mathbf{g}}$.

Lemma. *(1) If the double two-form* $\mathbf{A}$ *is symmetric with respect to its two pairs of antisymmetric indices, then* $E_{ij} = E_{ji}$*,* $D_{ij} = D_{ji}$*, and* $H_{ij} = B_{ji}$*. (2) If* $\mathbf{A}$ *is a symmetric double two-form such that*

$$A_{\alpha\beta} \equiv A^\lambda{}_{\alpha,\lambda\beta} = c g_{\alpha\beta}, \tag{18}$$

then $H_{ij} = H_{ji} = B_{ji} = B_{ij}$ *and* $E_{ij} = D_{ij}$.

Proof. (1) If $\mathbf{A}$ is a *symmetric* double two-form, the proof is immediate. (2) The Lanczos identity [8] for a symmetric double two-form, with a tilde representing the spacetime double dual, is given by

$$\tilde{A}_{\alpha\beta,\lambda\mu} + A_{\alpha\beta,\lambda\mu} \equiv C_{\alpha\lambda} g_{\beta\mu} - C_{\alpha\mu} g_{\beta\lambda} + C_{\beta\mu} g_{\alpha\lambda} - C_{\beta\lambda} g_{\alpha\mu}, \qquad (19)$$

with

$$\begin{aligned} C_{\alpha\beta} &\equiv A_{\alpha\beta} - \frac{1}{4} A g_{\alpha\beta}, \\ A &\equiv A^{\lambda\alpha}{}_{,\lambda\alpha}. \end{aligned} \qquad (20)$$

The equalities $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$ follow by a simple calculation that employs the relation $\eta_{0ijk} = N\eta_{ijk}$ between the spacetime and space volume forms.

In order to extend the treatment to the non-vacuum case and to avoid introducing unphysical characteristics in the solution of the Bianchi equations, we will keep as independent unknowns the four tensors $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$, which will not be regarded necessarily as symmetric. The symmetries will be imposed eventually on the initial data and shown to be conserved by evolution.

We now express the Bianchi equations in terms of the time dependent space tensors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$. We use the following relations found by inverting the equations (16) and (17)

$$\begin{aligned} A_{0i,0j} &= -N^2 E_{ij}, && (21) \\ A_{hk,0j} &= N\eta^i{}_{hk} H_{ij}, && (22) \\ A_{hk,lm} &= \eta^i{}_{hk} \eta^j{}_{lm} D_{ij}, && (23) \\ A_{0j,hk} &= N\eta^i{}_{hk} B_{ji}. && (24) \end{aligned}$$

We will express spacetime covariant derivatives ∇ of the spacetime tensor $\mathbf{A}$ in terms of space covariant derivatives $\bar{\nabla}$ and time derivatives $\hat{\partial}_0$ of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ by using the connection coefficients in 3+1 form as given in (7) and (8).

The first Bianchi equation (14) with $[\lambda\mu] = [0j]$ has the form

$$\nabla_0 A_{hk,0j} + \nabla_k A_{0h,0j} - \nabla_h A_{0k,0j} = 0. \qquad (25)$$

A calculation incorporating (7), (8) and (21)-(24), then grouping derivatives using $\hat{\partial}_0$ and $\bar{\nabla}_i$, yields the first Bianchi equation in the form

$$\hat{\partial}_0(\eta^i{}_{hk}H_{ij}) + 2N\bar{\nabla}_{[h}E_{k]j} + (L_1)_{hk,j} = 0, \tag{26}$$

$$\begin{aligned}(L_1)_{hk,j} \equiv\; & NK^l{}_j\eta^i{}_{hk}H_{il} + 2(\bar{\nabla}_{[h}N)E_{k]j} + 2N\eta^i{}_{lj}K^l{}_{[k}B_{h]i} \\ & -(\bar{\nabla}^l N)\eta^i{}_{hk}\eta^m{}_{lj}D_{im}.\end{aligned} \tag{27}$$

The second Bianchi equation (15), with $[\lambda\mu] = [0j]$, has the form

$$\nabla_0 A^0{}_{i,0j} + \nabla_h A^h{}_{i,0j} = J_{0ji}, \tag{28}$$

where $\mathbf{J}$ is zero in vacuum. A calculation similar to the one above yields for the second Bianchi equation

$$\hat{\partial}_0 E_{ij} - N\eta^{hl}{}_i\bar{\nabla}_h H_{lj} + (L_2)_{ij} = J_{0ji}, \tag{29}$$

$$\begin{aligned}(L_2)_{ij} \equiv\; & -N(\mathrm{tr}\mathbf{K})E_{ij} + NK^k{}_jE_{ik} + 2NK_i{}^kE_{kj} \\ & -(\bar{\nabla}_h N)\eta^{hl}{}_iH_{lj} + NK^k{}_h\eta^{lh}{}_i\eta^m{}_{kj}D_{lm} + (\bar{\nabla}^k N)\eta^l{}_{kj}B_{il}.\end{aligned} \tag{30}$$

The non-principal terms in the first two Bianchi equations (26) and (29) are linear in $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{H}$, as well as in the other geometrical elements $N\mathbf{K}$ and $\bar{\nabla}N$. The characteristic matrix of the principal terms is symmetrizable. The unknowns $E_{i(j)}$ and $H_{i(j)}$, with fixed j and $i = 1, 2, 3$ appear only in the equations with given j. The other unknowns appear in non-principal terms. The characteristic matrix is composed of three blocks around the diagonal, each corresponding to one given j.

The j^{th} block of the characteristic matrix in an orthonormal frame for the space metric $\bar{\mathbf{g}}$, with unknowns listed horizontally and equations listed vertically, (j is suppressed) is given by

$$\begin{array}{c} \\ (29)_1 \\ (29)_2 \\ (29)_3 \\ (26)_{23} \\ (26)_{31} \\ (26)_{12} \end{array}
\begin{array}{c} \begin{array}{cccccc} E_1 & E_2 & E_3 & H_1 & H_2 & H_3 \end{array} \\
\begin{pmatrix}
\xi_0 & 0 & 0 & 0 & N\xi_3 & -N\xi_2 \\
0 & \xi_0 & 0 & -N\xi_3 & 0 & N\xi_1 \\
0 & 0 & \xi_0 & N\xi_2 & -N\xi_1 & 0 \\
0 & -N\xi_3 & N\xi_2 & \xi_0 & 0 & 0 \\
N\xi_3 & 0 & -N\xi_1 & 0 & \xi_0 & 0 \\
-N\xi_2 & N\xi_1 & 0 & 0 & 0 & \xi_0
\end{pmatrix}\end{array}. \tag{31}$$

This matrix is symmetric and its determinant is the characteristic polynomial of the $\mathbf{E}$, $\mathbf{H}$ system. It is given by

$$-N^6(\xi_0\xi^0)(\xi_\alpha\xi^\alpha)^2. \tag{32}$$

The characteristic matrix is symmetric in an orthonormal space frame and the timelike direction defined by $\hat{\partial}_0$ has a coefficient matrix T_0 that is positive definite (here T_0 is the unit matrix). Therefore, the first order system is symmetrizable hyperbolic. We do not have to compute the symmetrized form explicitly because we will obtain the energy estimate directly by using the contravariant associates E^{ij}, $H^{ij}, \ldots$ of the unknowns.

The second pair of Bianchi equations is obtained from (14) and (15) with $[\lambda\mu] = [lm]$. We obtain from (14)

$$\hat{\partial}_0(\eta^i{}_{hk}\eta^j{}_{lm}D_{ij}) + 2N\eta^j{}_{lm}\bar{\nabla}_{[k}B_{h]j} + (L_3)_{hk,lm} = 0, \tag{33}$$

$$\begin{aligned}(L_3)_{hk,lm} \equiv\ & 2N\eta^n{}_{j[m}K^j{}_{l]}\eta^i{}_{hk}D_{in} + 2\eta^j{}_{lm}(\bar{\nabla}_{[k}N)B_{h]j} \\ & +2NK_{l[h}E_{k]m} + 2NK_{m[k}E_{h]l} + 2H_{i[l}(\bar{\nabla}_{m]}N)\eta^i{}_{hk}.\end{aligned} \tag{34}$$

Analogously, from (15) we obtain

$$\hat{\partial}_0(\eta^j{}_{lm}B_{ij}) - N\eta^{kh}{}_i\eta^j{}_{lm}\bar{\nabla}_h D_{kj} + (L_4)_{i,lm} = -NJ_{lmi}, \tag{35}$$

$$\begin{aligned}(L_4)_{i,lm} \equiv\ & -N(\mathrm{tr}\mathbf{K})\eta^j{}_{lm}B_{ij} + 2N\eta^h{}_{j[m}K^j{}_{l]}B_{ih} + 2NK^h{}_i\eta^j{}_{lm}B_{hj} \\ & -(\bar{\nabla}_j N)\eta^{hj}{}_i\eta^n{}_{lm}D_{hn} - 2N\eta^j{}_{hi}H_{j[m}K^h{}_{l]} + 2E_{i[m}\bar{\nabla}_{l]}N.\end{aligned} \tag{36}$$

Consider the system (33) and (35) with $[lm]$ fixed. Then j in η_{jlm} is also fixed. The characteristic matrix for the $[lm]$ equations, with unknowns D_{ij} and B_{ij}, j fixed, with an orthonormal space frame, is the same as the matrix (31).

If the spacetime metric $\mathbf{g}$ is considered as given, as well as the sources, the Bianchi equations (26), (29), (33), and (35) form a linear symmetric hyperbolic system with domain of dependence determined by the light cone of $\mathbf{g}$. The coefficients of the terms of order zero are $\bar{\nabla}N$ or $N\mathbf{K}$. The system is homogeneous in vacuum (zero sources).

4 Bel Energy in a Strip

Multiply (26) by $\frac{1}{2}\eta_l{}^{hk}H^{lj}$ and recall that $\eta_l{}^{hk}\eta^i{}_{hk} = 2\delta^i{}_l$, $\eta_{lrk}\eta^{ihk} = \delta^i{}_l\delta^h{}_r - \delta^i{}_r\delta^h{}_l$, and $\hat{\partial}_0 g^{ij} = 2NK^{ij}$. Then we find that

$$\frac{1}{2}\eta_l{}^{hk}H^{lj}\hat{\partial}_0(\eta^i{}_{hk}H_{ij}) = \frac{1}{2}\hat{\partial}_0(H_{ij}H^{ij}) - M_1, \tag{37}$$

$$\begin{aligned} M_1 &\equiv \frac{1}{4}\eta^l{}_{rs}H_{lm}\eta^i{}_{hk}H_{ij}\hat{\partial}_0(g^{hr}g^{ks}g^{jm}) \\ &= N((\mathrm{tr}\mathbf{K})H^{ij} - K^i{}_lH^{lj} + K_l{}^jH^{il})H_{ij}. \end{aligned} \tag{38}$$

Likewise, multiply (29) by E^{ij} to obtain

$$E^{ij}\hat{\partial}_0 E_{ij} = \frac{1}{2}\hat{\partial}_0(E_{ij}E^{ij}) - M_2, \tag{39}$$

$$M_2 \equiv N(K^i{}_lE^{lj} + K_l{}^jE^{il})E_{ij}. \tag{40}$$

Multiply (33) by $(1/4)\eta_r{}^{hk}\eta_s{}^{lm}D^{rs}$ and (35) by $(1/2)\eta_h{}^{lm}B^{ih}$ to obtain analogous results for the second pair of Bianchi equations. Sum the expressions so obtained from the four Bianchi equations (26), (29), (33), and (35). The spatial derivatives add to form an exact spatial divergence, just as for all symmetric systems. Indeed, we obtain

$$\begin{aligned} &\frac{1}{2}\hat{\partial}_0(|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{D}|^2 + |\mathbf{B}|^2) + \bar{\nabla}_h(NE^{ij}\eta^{lh}{}_iH_{lj}) \\ &\qquad -\bar{\nabla}_h(NB^{ij}\eta^{lh}{}_iD_{lj}) = Q(\mathbf{E},\mathbf{H},\mathbf{D},\mathbf{B}) + \mathcal{S}, \end{aligned} \tag{41}$$

where we have denoted by $|\cdot|$ the pointwise $\bar{\mathbf{g}}$ norm of a space tensor, and where Q is a quadratic form with coefficients $\bar{\nabla}N$ and $N\mathbf{K}$ given by

$$\begin{aligned} Q \equiv\ & -\frac{1}{2}\eta_l{}^{hk}H^{lj}(L_1)_{hk,j} - E^{ij}(L_2)_{ij} - \frac{1}{4}\eta_r{}^{hk}\eta_s{}^{lm}D^{rs}(L_3)_{hk,lm} \\ & -\frac{1}{2}\eta_h{}^{lm}B^{ih}(L_4)_{i,lm} + (\bar{\nabla}_hN)E^{ij}\eta^{lh}{}_iH_{lj} - (\bar{\nabla}_hN)B^{ij}\eta^{lh}{}_iD_{lj} \\ & +M_1 + M_2 + M_3 + M_4. \end{aligned} \tag{42}$$

The source term $\mathcal{S}$, zero in vacuum, is

$$\mathcal{S} \equiv J_{0ij}E^{ij} - \frac{1}{2}NJ_{lmi}\eta_h{}^{lm}B^{ih}. \tag{43}$$

We define the *Bel energy* at time t of the field $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B})$, called a "Bianchi field" when it satisfies the Bianchi equations, to be the integral

$$\mathcal{B}(t) \equiv \frac{1}{2} \int_{M_t} (|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{D}|^2 + |\mathbf{B}|^2) \mu_{\bar{\mathbf{g}}_t}. \tag{44}$$

We prove the following

Theorem 1. *Suppose that* $\mathbf{g}$ *is* $\mathcal{C}^1$ *on* $M \times [0,T]$ *and that the* $\bar{\mathbf{g}}_t$ *norms of* $\bar{\nabla} N$ *and* $N\mathbf{K}$ *are uniformly bounded on* M_t, $t \in [0,T]$. *Denote by* $\pi(t)$ *the supremum*

$$\pi(t) = \mathrm{Sup}_{M_t}(|\bar{\nabla} N| + |N\mathbf{K}|). \tag{45}$$

Suppose the matter source $\mathbf{J} \in L^1([0,T], L^2(M_t))$, *then the Bel energy of a* $\mathcal{C}^1$ *Bianchi field with compact support in space satisfies for* $0 \le t \le T$ *the following inequality*

$$\mathcal{B}(t) \le \left\{ \mathcal{B}(0) + \int_0^t \|\mathcal{S}\|_{L^2(M_\tau)} d\tau \right\} \exp(C \int_0^t \pi(\tau) d\tau), \tag{46}$$

where C *is a given positive number.*

Proof. We integrate the identity (41) on the strip $M \times [0,t]$ with respect to the volume element $\mu_{\bar{\mathbf{g}}_\tau} d\tau$. If the Bianchi field has support compact in space, the integral of the space divergence term vanishes. The integration of a function $\hat{\partial}_0 F$ on a strip of spacetime with respect to the volume form $d\tau \, \mu_{\bar{\mathbf{g}}_\tau}$ goes as follows

$$\begin{aligned}
\int_0^t \int_{M_\tau} \hat{\partial}_0 F \, d\tau \, \mu_{\bar{\mathbf{g}}_\tau} &\equiv \int_0^t \int_{M_\tau} (\partial_t - \beta^i \partial_i) F \, d\tau \, \mu_{\bar{\mathbf{g}}_\tau} \\
&= \int_0^t \int_{M_\tau} \partial_\tau (F (\det \bar{\mathbf{g}}_\tau)^{1/2}) \, d\tau \, d^3 x \\
&\quad - \int_0^t \int_{M_\tau} \left(F (\det \bar{\mathbf{g}}_\tau)^{-1/2} \partial_\tau (\det \bar{\mathbf{g}}_\tau)^{1/2} \right) \mu_{\bar{\mathbf{g}}_\tau} \, d\tau \\
&\quad - \int_0^t \int_{M_\tau} \bar{\nabla}_i (\beta^i F) \, \mu_{\bar{\mathbf{g}}_\tau} \, d\tau + \int_0^t \int_{M_\tau} F \bar{\nabla}_i \beta^i \, \mu_{\bar{\mathbf{g}}_\tau} \, d\tau.
\end{aligned} \tag{47}$$

Therefore, if F has compact support in space, we can express the right hand side of (47) as

$$\int_{M_\tau} F \, \mu_{\bar{\mathbf{g}}_\tau} - \int_{M_0} F \, \mu_{\bar{\mathbf{g}}_0} + \int_0^t \int_{M_\tau} N (\mathrm{tr} \mathbf{K}) F \, \mu_{\bar{\mathbf{g}}_\tau} \, d\tau, \tag{48}$$

where we have used

$$\partial_\tau (\det \bar{\mathbf{g}}_\tau)^{1/2} = -(\det \bar{\mathbf{g}}_\tau)^{1/2} N(\mathrm{tr}\mathbf{K}) + (\det \bar{\mathbf{g}}_\tau)^{1/2} \bar{\nabla}_i \beta^i \tag{49}$$

with $\mathrm{tr}\mathbf{K} = K^j{}_j$.

The integration of (41) on a strip leads therefore to the equality

$$\mathcal{B}(t) = \mathcal{B}(0) + \int_0^t \int_{M_\tau} (\tilde{Q} + \mathcal{S})\, \mu_{\bar{\mathbf{g}}_\tau}\, d\tau, \tag{50}$$

with

$$\tilde{Q} = Q + \frac{1}{2} N(\mathrm{tr}\mathbf{K})(|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{D}|^2 + |\mathbf{B}|^2).$$

We deduce from this equality and the expression (42) for Q the following inequality, with C some number

$$\mathcal{B}(t) \le \mathcal{B}(0) + C \int_0^t \mathrm{Sup}_{M_\tau}(|\bar{\nabla} N| + |N\mathbf{K}|)\mathcal{B}(\tau)\, d\tau + \int_0^t \|\mathcal{S}\|_{L^2(M_\tau)}\, d\tau. \tag{51}$$

This inequality and the resolution of the corresponding equality imply the result.

Corollary. *If the metrics $\bar{\mathbf{g}}_t$ are complete, the inequality given in the theorem extends to $\mathcal{C}^1$ Bianchi fields that are square integrable in $\mu_{\bar{\mathbf{g}}_t}$.*

Proof. One uses a truncating sequence.

Remark 1. The quantities $-\bar{\nabla}_k N$ and $-2K_{ij}$ are, respectively, the $(0k)$ and (ij) components of the Lie derivative $\mathcal{L}_{\mathbf{n}}\mathbf{g}$ of the spacetime metric $\mathbf{g}$ with respect to the unit normal $\mathbf{n}$ of M_t. [Its (00) component is identically zero.] The Bel energy, therefore, is conserved if this Lie derivative is zero.

Remark 2. Energy estimates for the double two-form $\mathbf{A}$ can be deduced from the Bel-Robinson tensor of $\mathbf{A}$ when $\mathbf{A}$ is assumed to be symmetric in its pairs of indices and when a timelike vector field has been chosen[9]. However, we do not make such a hypothesis about the symmetry of $\mathbf{A}$ here.

5 Local Energy Estimate

We take as a domain Ω of spacetime the closure of a connected open set whose boundary $\partial\Omega$ consists of three parts: a domain ω_t of M_t, a domain ω_0 of M_0,

and a lateral boundary L. We assume L is spacelike or null and "ingoing," that is, timelike lines entering Ω at a point of L are past directed. We also assume that the boundary $\partial\Omega$ is regular in the sense of Stoke's formula. We use the identity (41) previously found and integrate this identity on Ω with respect to the volume form $\mu_{\bar{\mathbf{g}}_\tau}\, d\tau$. Let $f(t,x) = t+\phi(x)$ be the local equation of L. Set $\nu_i = \partial_i f = \partial_i \phi$ and $\nu_0 = 1 - \beta^i \nu_i$; these are the components in the coframe θ^α of the spacetime gradient $\partial_\alpha f = \nu_\alpha$. Then we have from (41)

$$\begin{aligned}\frac{1}{2}\int_{\omega_t}(|\mathbf{E}|^2+|\mathbf{H}|^2+|\mathbf{D}|^2+|\mathbf{B}|^2)\,\mu_{\bar{\mathbf{g}}_t} - \frac{1}{2}\int_{\omega_0}(|\mathbf{E}|^2+|\mathbf{H}|^2+|\mathbf{D}|^2+|\mathbf{B}|^2)\,\mu_{\bar{\mathbf{g}}_0}\\ +\int_L\Big[\frac{1}{2}\nu_0(|\mathbf{E}|^2+|\mathbf{H}|^2+|\mathbf{D}|^2+|\mathbf{B}|^2)+N\nu_h\{(\mathbf{E}\wedge\mathbf{H})^h-(\mathbf{B}\wedge\mathbf{D})^h\}\Big]\mu_{\bar{\mathbf{g}}} \;=\\ =\int_0^t\int_{\omega_\tau}(\tilde{Q}+\mathcal{S})\,\mu_{\bar{\mathbf{g}}_\tau}\,d\tau. \qquad (52)\end{aligned}$$

We have set

$$(\mathbf{E}\wedge\mathbf{H})^h = \eta^{lh}{}_i E^{ij} H_{lj} \equiv \sum_j (\mathbf{E}\wedge\mathbf{H})^h_{(j)},$$

that is, $(\mathbf{E}\wedge\mathbf{H})^h_{(j)}$, for each fixed j, is the vector product in three dimensions of the vectors $\vec{E}^{(j)}$ and $\vec{H}_{(j)}$. Therefore, the $\bar{\mathbf{g}}$ norm of the vector $(\mathbf{E}\wedge\mathbf{H})^h$ satisfies

$$|\mathbf{E}\wedge\mathbf{H}| \le \sum_j |\vec{E}^{(j)}\wedge\vec{H}_{(j)}| \le \sum_j |\vec{E}^{(j)}||\vec{H}_{(j)}| \le \frac{1}{2}(|\mathbf{E}|^2+|\mathbf{H}|^2). \qquad (53)$$

Let $\bar{\nu} = (\nu_i)$, $\nu_0 > 0$, and $|\bar{\nu}|_{\bar{\mathbf{g}}} \le N^{-1}\nu_0$, that is, let $\underline{\nu} = (\nu_0, \nu_i)$ be timelike or null, as it must be for L. [Note that our sign conventions for $\underline{\nu}$ are suitable for applications of Stokes's theorem in a spacetime with signature $(-+++)$.] We can now deduce from (53) and an analogous inequality for $(\mathbf{D},\mathbf{B})$ that the integral over L in (52) is non-negative. It is strictly positive if $\underline{\nu}$ is timelike.

Remark. The integral over a null boundary L is also strictly positive if (53) is a strict inequality. This will be the case if any of the norms of the vector products of $\vec{E}^{(j)}$ by $\vec{H}_{(j)}$ is less than the product of the norms of these vectors, that is, if any of these pairs of vectors are non-orthogonal: $E^{i(j)}H_{i(j)} \neq 0$ or $D^{i(j)}B_{i(j)} \neq 0$ for some j. An analogous property has been obtained through the use of the Bel-Robinson tensor of the Weyl curvature in [10].

We define $\mathcal{B}(\omega_t)$ by replacing in the definition of $\mathcal{B}(t)$ the integral over M_t by an integral over ω_t

$$\mathcal{B}(\omega_t) = \frac{1}{2}\int_{\omega_t} (|\mathbf{E}|^2 + |\mathbf{H}|^2 + |\mathbf{D}|^2 + |\mathbf{B}|^2)\,\mu_{\bar{\mathbf{g}}_t}. \tag{54}$$

A result of the non-negativity of the integral on L is that the inequality (51) holds on Ω with $\mathcal{B}(\tau)$ replaced by $\mathcal{B}(\omega_\tau)$, $\omega_\tau \equiv M_\tau \cap \Omega$, and M_τ replaced by ω_τ. Therefore the inequality (46) also holds with the same replacements. In particular, we have $\mathcal{B}(\omega_\tau) = 0$ if $\mathcal{B}(\omega_0) = 0$ and $\mathbf{J} = 0$ (vacuum case). Then $\mathbf{E} = \mathbf{H} = \mathbf{D} = \mathbf{B} = 0$ in Ω if they vanish on the intersection of M_0 with the past of Ω. (Cf. for related results [10] and [11].) Note that such a result is not sufficient to prove the propagation of gravitation with the speed of light because it treats only curvature tensors that are zero in some domain, not the difference of non-zero curvature tensors. The Bianchi equations are not by themselves sufficient to estimate such differences because their coefficients depend on the metric, which itself depends on the curvature.

In the next section, we will give a further first order symmetric hyperbolic system linking the metric and the connection to our Bianchi field. This further system is inspired by an analogous one constructed in conjunction with the Weyl tensor by Friedrich [2].

6 Determination of $(\bar{\mathbf{\Gamma}}, \mathbf{K})$ from Knowledge of the Bianchi Fields

We will need the $3+1$ decomposition of the Riemann tensor, which is found by combining (5), (7), and (8); namely,

$$R_{ij,kl} = \bar{R}_{ij,kl} + 2K_{i[k}K_{l]j}, \tag{55}$$

$$R_{0i,jk} = 2N\bar{\nabla}_{[j}K_{k]i}, \tag{56}$$

$$R_{0i,0j} = N(\hat{\partial}_0 K_{ij} + NK_{ik}K^k{}_j + \bar{\nabla}_i\partial_j N). \tag{57}$$

From these formulae one obtains the following ones for the Ricci curvature $R_{\alpha\beta} = R^\gamma{}_{\alpha,\gamma\beta}$

$$R_{ij} = \bar{R}_{ij} - N^{-1}\hat{\partial}_0 K_{ij} + K_{ij}\mathrm{tr}\mathbf{K} - 2K_{ik}K^k{}_j - N^{-1}\bar{\nabla}_i\partial_j N, \tag{58}$$

$$R_{0j} = N(\partial_j \mathrm{tr}\mathbf{K} - \bar{\nabla}_h K^h{}_j), \tag{59}$$

$$R_{00} = N(\hat{\partial}_0 \mathrm{tr}\mathbf{K} - NK_{ij}K^{ij} + \bar{\nabla}^i\partial_i N). \tag{60}$$

The identity

$$\hat{\partial}_0 g_{ij} \equiv -2NK_{ij} \tag{61}$$

and the expression for the spatial Christoffel symbols give

$$\hat{\partial}_0 \bar{\Gamma}^h{}_{ij} \equiv \bar{\nabla}^h (NK_{ij}) - 2\bar{\nabla}_{(i}(NK_{j)}{}^h). \tag{62}$$

Therefore, from the identity (56), we obtain the identity

$$\hat{\partial}_0 \bar{\Gamma}^h{}_{ij} + N\bar{\nabla}^h K_{ij} = K_{ij}\partial^h N - 2K^h{}_{(i}\partial_{j)}N - 2R_{0(i,j)}{}^h. \tag{63}$$

On the other hand, the identities (57) and (58) imply the identity

$$\hat{\partial}_0 K_{ij} + N\bar{R}_{ij} + \bar{\nabla}_j \partial_i N \equiv -2NR^0{}_{i,0j} - N(\mathrm{tr}\mathbf{K})K_{ij} + NR_{ij}. \tag{64}$$

We obtain equations relating $\bar{\mathbf{\Gamma}}$ and $\mathbf{K}$ to a double two-form $\mathbf{A}$ and matter sources by replacing, in the identities (63) and (64), $R_{0(i,j)}{}^h$ by $(A_{0(i,j)}{}^h + A^h{}_{(j,i)0})/2$, $R^0{}_{i,0j}$ by $(A^0{}_{i,0j} + A^0{}_{j,0i})/2$, and the Ricci tensor of spacetime by a given tensor $\boldsymbol{\rho}$, zero in vacuum. The terms involving $\mathbf{A}$ are then replaced by Bianchi fields as in (21)-(24).

The first set of identities (63) leads to equations with principal terms

$$\hat{\partial}_0 \bar{\Gamma}^h{}_{ij} + Ng^{hk}\partial_k K_{ij}. \tag{65}$$

To deduce from the second identity (64) equations which will form together with the previous ones a symmetric hyperbolic system, we set

$$N = \alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}, \tag{66}$$

where α is a given scalar density of weight one. This is the "algebraic gauge" [12]. (It can also be considered as a change in the name of the unknown from N to α [14].) The condition (66), if $\bar{\mathbf{\Gamma}}$ denotes the Christoffel symbols of $\bar{\mathbf{g}}$, implies that

$$\bar{\Gamma}^h{}_{ih} = \partial_i \log N + \partial_i \log \alpha. \tag{67}$$

The second set of identities (64) now yields the following equations, where N denotes $\alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$,

$$\begin{aligned} \hat{\partial}_0 K_{ij} + N\partial_h \bar{\Gamma}^h{}_{ij} \;&=\; N[\bar{\Gamma}^m{}_{ih}\bar{\Gamma}^h{}_{jm} - (\bar{\Gamma}^h{}_{ih} - \partial_i \log\alpha)(\bar{\Gamma}^k{}_{jk} - \partial_j \log\alpha)] \\ &+N(\partial_i\partial_j \log\alpha - \bar{\Gamma}^k{}_{ij}\partial_k \log\alpha) - N(E_{ij} + E_{ji}) - N(\mathrm{tr}\mathbf{K})K_{ij} + N\rho_{ij}. \end{aligned} \tag{68}$$

The first set (63) yields

$$\hat{\partial}_0 \bar{\Gamma}^h{}_{ij} + N \bar{\nabla}^h K_{ij} = N K_{ij} g^{hk} (\bar{\Gamma}^m{}_{mk} - \partial_k \log \alpha) \\ -2N K^h{}_{(i} (\bar{\Gamma}^m{}_{j)m} - \partial_{j)} \log \alpha) - N \eta^k{}_{(i}{}^h B_{j)k} - N H_{k(j} \eta^k{}_{i)}{}^h. \tag{69}$$

We see from the principal parts of (68) and (69) that the system obtained for $\mathbf{K}$ and $\bar{\mathbf{\Gamma}}$ has a characteristic matrix composed of six blocks around the diagonal, each block a four-by-four matrix that is symmetrizable hyperbolic with characteristic polynomial $N^4(\xi_0 \xi^0)(\xi_\alpha \xi^\alpha)$. The characteristic matrix in a spatial orthonormal frame has blocks of the form

$$\begin{pmatrix} \xi_0 & N\xi_1 & N\xi_2 & N\xi_3 \\ N\xi_1 & \xi_0 & 0 & 0 \\ N\xi_2 & 0 & \xi_0 & 0 \\ N\xi_3 & 0 & 0 & \xi_0 \end{pmatrix}. \tag{70}$$

Remark. In the equations considered above, N is to be replaced by $\alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$. Instead of this algebraic replacement, we can consider an evolution equation for N of the type

$$\hat{\partial}_0 N + N^2 \mathrm{tr}\mathbf{K} = N f, \tag{71}$$

where f is an arbitrary function on spacetime. For $f = 0$, (71) is the harmonic slicing condition. For arbitrary f, the general solution of (71), if $\bar{\mathbf{g}}$ and $\mathbf{K}$ are linked as in (61), is

$$N = \gamma^{-1}(\det \bar{\mathbf{g}})^{1/2}, \tag{72}$$

with γ such that

$$\hat{\partial}_0 \gamma + f\gamma = 0, \tag{73}$$

that is, γ satisfies a linear first order differential equation depending only on the known quantities β and f. It does not depend on any of the previously defined unknowns $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \bar{\mathbf{g}}, \mathbf{K}, \bar{\mathbf{\Gamma}})$ of our system in algebraic gauge. The modified system consists of the equations in algebraic gauge [(26), (29), (33), (35), (61), (68), (69)] but with N now replaced by (72). Therefore, α is replaced by γ. The additional equation is (71), or, on account of (72), this role can be regarded as played by (73).

7 Symmetric Hyperbolic System for $\mathbf{u} \equiv (\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \bar{\mathbf{g}}, \mathbf{K}, \bar{\mathbf{\Gamma}})$

We denote by $\mathbf{S}$ the system composed of the equations (26), (29), (33), (35), (61), (68), and (69), where the lapse function N is replaced by $\alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$. This system is satisfied by solutions of the Einstein equations whose shift $\boldsymbol{\beta}$, hidden in the operator $\hat{\partial}_0$, has the given arbitrary values and whose lapse has the form $N = \alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$. (Clearly, any $N > 0$ can be written in this form.) The following lemma results from the previous paragraphs.

Lemma. *For arbitrary α and $\boldsymbol{\beta}$, and given matter sources $\boldsymbol{\rho}$, the system $\mathbf{S}$ is a first order symmetric hyperbolic system for the unknowns $\mathbf{u}$.*

Note that in this Lemma, the various elements $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$, $\bar{\mathbf{g}}$, $\mathbf{K}$, and $\bar{\mathbf{\Gamma}}$ are considered as independent. For example, *a priori*, we neither know that $\bar{\mathbf{\Gamma}}$ denotes the Christoffel symbols of $\bar{\mathbf{g}}$ nor that $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ are identified with components of the Riemann tensor of spacetime.

An analogous Lemma holds for the system $\mathbf{S}'$ that has an additional unknown N (equivalently, γ) and an additional equation (71) [equivalently, (73)] with f replacing α as an arbitrary function.

We now consider the *vacuum* case.

Initial Values. The original Cauchy data for the Einstein equations are, with $\boldsymbol{\phi}$ a properly Riemannian metric and $\boldsymbol{\psi}$ a second rank tensor on M_0,

$$\bar{\mathbf{g}}|_0 = \boldsymbol{\phi}, \quad \mathbf{K}|_0 = \boldsymbol{\psi}. \tag{74}$$

The tensors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ must satisfy the constraints, which read in vacuum,

$$R_{0i} = 0, \tag{75}$$

$$\begin{aligned} 0 = G_{00} &\equiv R_{00} - \frac{1}{2} g_{00} R \\ &= R_{00} + \frac{1}{2} N^2 g^{\alpha\beta} R_{\alpha\beta} \\ &= \frac{1}{2} N^2 (\bar{R} - K_{ij} K^{ij} + (\mathrm{tr}\mathbf{K})^2), \end{aligned} \tag{76}$$

with $\bar{R} = g^{ij} \bar{R}_{ij}$. The initial data given by $\boldsymbol{\phi}$ determine the Cauchy data $\bar{\Gamma}^h{}_{ij}|_0$ and thus $\bar{R}_{ij,kl}|_0$. Then, $R_{ij,kl}|_0$ is determined by using also $\boldsymbol{\psi}$. To

determine the initial values of the other components of the unknown $\mathbf{u}$ of the system $\mathbf{S}$, we use the arbitrarily given data α and $\boldsymbol{\beta}$. In particular, we use $N = \alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$ to find $R_{0i,jk}|_0$ and to compute $\bar{\nabla}_j \partial_i N|_0$ appearing in the identity (58). We deduce from (58) $\hat{\partial}_0 K_{ij}|_0$ when $R_{ij} = 0$, which enables $R_{0i,0j}$ to be found from (57). All of the components of the Riemann tensor of spacetime are then known on M_0. We identify them with the corresponding components of the double two-form $\mathbf{A}$ on M_0: the latter have thus *initially* the same symmetries as the Riemann tensor. We find the initial values of $(\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B})$ according to their definitions in terms of $\mathbf{A}$.

8 Existence Theorems

In order to define Sobolev spaces on a manifold M, it is convenient to endow M with a $\mathcal{C}^\infty$ Riemannian metric $\bar{\boldsymbol{e}}$. We will suppose that $\bar{\boldsymbol{e}}$ has a non-zero injectivity radius, $\delta_0 > 0$, hence is complete, and that it has Riemannian curvature uniformly bounded on M, as well as its derivatives of all orders relevant for the required Sobolev properties to hold.

With I an interval of R, and under the hypotheses made on $\bar{\boldsymbol{e}}$, each manifold $M \times I$ endowed with the metric $\bar{\boldsymbol{e}} - dt^2$ is globally hyperbolic [15, 13]. We denote by D the covariant derivative associated with $\bar{\boldsymbol{e}}$ (the "$\bar{\boldsymbol{e}}$-covariant derivative").

Theorem 2. *Hypotheses. a. The arbitrary quantities are such that, with I an interval containing 0, α is continuous on $M \times I$, and there exist numbers $A_1,\ A_2 > 0$ such that on $M \times I$*

$$0 < A_1 \le \alpha \le A_2,$$

$$\mathbf{D}\alpha, \boldsymbol{\beta} \in \mathcal{C}^0(I, H_{s+1}).$$

b. The initial data are such that

$$\mathbf{D}\boldsymbol{\phi}, \boldsymbol{\psi} \in H_{s+1},$$

and $\boldsymbol{\phi}$ is a continuous properly Riemannian metric on M uniformly equivalent to $\bar{\boldsymbol{e}}$. That is, there exist strictly positive numbers b_1, b_2 such that at each point of M and for any tangent vector ξ to M at this point,

$$b_1 \bar{\boldsymbol{e}}(\boldsymbol{\xi}, \boldsymbol{\xi}) \le \boldsymbol{\phi}(\boldsymbol{\xi}, \boldsymbol{\xi}) \le b_2 \bar{\boldsymbol{e}}(\boldsymbol{\xi}, \boldsymbol{\xi}).$$

Conclusion. If $s \geq 3$, there exists a number $T > 0$, $[0,T] \in I$, such that the system $\mathbf{S}$ has one and only one $\mathcal{C}^1$ solution $\mathbf{u}$ on $[0,T] \times M$ taking the Cauchy data $\mathbf{u}_0$ deduced from $\boldsymbol{\phi}$, $\boldsymbol{\psi}, \alpha, \boldsymbol{\beta}$. The solution $\mathbf{u}$ is such that the components of $\mathbf{u}$ different from $\bar{\mathbf{g}}$, as well as $\bar{\mathbf{g}} - \boldsymbol{\phi}$, are in $\mathcal{C}^0([0,T], H_s) \cap \mathcal{C}^1([0,T], H_{s-1})$. There exist B_1 and B_2 such that

$$B_1 \bar{\boldsymbol{e}}(\boldsymbol{\xi}, \boldsymbol{\xi}) \leq \bar{\mathbf{g}}(\boldsymbol{\xi}, \boldsymbol{\xi}) \leq B_2 \bar{\boldsymbol{e}}(\boldsymbol{\xi}, \boldsymbol{\xi}).$$

Proof. 1. We replace the equation (61) in the system $\mathbf{S}$ by the equation

$$\hat{\partial}_0 (g_{ij} - \phi_{ij}) = -2NK_{ij} + \mathcal{L}_\beta \phi_{ij}. \tag{77}$$

We still have a symmetric first order system $\tilde{\mathbf{S}}$ for the unknown $\tilde{\mathbf{u}}$, deduced from $\mathbf{u}$ by replacing $\bar{\mathbf{g}}$ by $\bar{\mathbf{g}} - \boldsymbol{\phi}$. This quasilinear system is hyperbolic as long as $\bar{\mathbf{g}}$ is properly Riemannian.

The hypotheses made on the arbitrary data α, $\boldsymbol{\beta}$ and on the initial data $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ imply, through the Sobolev multiplication properties, that the initial data $\tilde{\mathbf{u}}|_0$ belong to H_s whenever $s \geq 1$.

2. If the Riemannian manifold $(M, \bar{\boldsymbol{e}})$ is the Euclidean space R^3, the existence theorem is known, with $s > 3/2+1$. [Linear case: Friedrichs [16]. Quasilinear case: Fischer-Marsden [17], using semigroup methods, and Majda [18], using energy estimates.]

A proof under the hypotheses we have made on $(M, \bar{\boldsymbol{e}})$ and on the data can be obtained along the same lines, using energy estimates, because the spaces H_s on $(M, \bar{\boldsymbol{e}})$ have the same functional properties that they have on R^3. The energy inequalities for $\tilde{\mathbf{u}}$ and its spatially covariant $\bar{\boldsymbol{e}}$-derivatives of order $\leq s$ can be obtained directly on $[0,T] \times M$, as we have indicated in the case of the Bianchi equations. One uses s-energy estimates for linear symmetric first order hyperbolic systems and the Gårding duality method [19] to prove existence and uniqueness for the linearized system. One uses the contraction mapping principle in the norm $\mathcal{C}^0([0,T], H_{s-1})$ on the one hand, and boundedness in the norm $\mathcal{C}^0([0,T], H_s) \cap \mathcal{C}^1([0,T], H_{s-1})$ on the other hand, as in Majda [18], to prove the theorem for our quasilinear system.

Remark. The spacetime defined by the $\bar{\mathbf{g}}$ component of $\mathbf{u}$ together with α and $\boldsymbol{\beta}$ on $M \times [0, T]$ is globally hyperbolic, with a $\mathcal{C}^1$ metric if α and $\boldsymbol{\beta}$ are $\mathcal{C}^1$.

Domain of dependence. The exterior sheet of the characteristic cone of the system $\mathbf{S}$ at a point of spacetime V is the light cone of the spacetime metric $\mathbf{g}$. It has been proven by Leray [15], for globally hyperbolic Leray systems on a manifold, that the values of a solution of the Cauchy problem in a compact set $\bar{\omega}_t \subset M_t$ depend only on the values of the Cauchy data in the compact set $\bar{\omega}_0$ that is the intersection of M_0 with the past of $\bar{\omega}_t$. We will prove this result for our symmetric hyperbolic system.

We recall that on a spacetime $(V, \mathbf{g})$, $V = M \times R$, temporally oriented by the orientation of R, the past $\mathcal{P}(x)$ of a point x is the union of the paths timelike with respect to $\mathbf{g}$ and ending at x. The past $\mathcal{P}(\omega)$ of a subset $\omega \subset V$ is the union of the past of its points.

Lemma. *Let $(V, \mathbf{g})$ be a globally hyperbolic manifold with $V = M \times R$ and $\mathbf{g}$ a $\mathcal{C}^1$ metric. Let $\bar{\omega}_T$ be a compact subset of M_T with non-empty interior ω_T and boundary $\partial\omega_T$. Then,*

$$\bar{\Omega} = \mathcal{P}(\bar{\omega}_T) \cap \{t \geq 0\}$$

is a compact subset of V. Its boundary is

$$\partial\Omega \equiv \bar{\omega}_T \cup \bar{\omega}_0 \cup L.$$

The lateral boundary L is generated by null geodesic arcs issuing from points of $\partial\omega_T$. At an interior point of such an arc, L has a null tangent hyperplane and is ingoing.

Proof. On a globally hyperbolic manifold, the past of a closed set is closed; and the intersection of the past of a compact set with a "past compact set" (here, $t \geq 0$) is compact. Therefore, $\bar{\Omega}$ is compact.

If $y \in \bar{\Omega} \cap \{0 < t < T\}$, there exists $x \in \bar{\omega}_T$ such that $y \in \mathcal{P}(x)$. Suppose that the path xy is strictly timelike. Then there is an open neighborhood $V(y)$ of y in $\mathcal{P}(x)$ such that $z \in V(y)$ implies that $z \in \mathcal{P}(x)$. Consequently, y cannot be on the boundary L, which must therefore be generated by null paths. Analogous reasoning shows that x cannot be an interior point of ω_T. For the proof of the rest of this Lemma, see Leray [15], Lemma 97.

Recognizing that in a $\mathcal{C}^1$ globally hyperbolic manifold the lateral boundary L admits a null tangent hyperplane almost everywhere, we prove the following theorem.

Theorem 3. *(Domain of Dependence) Two $\mathcal{C}^1$ solutions $\mathbf{u}_1$ and $\mathbf{u}_2$ of $\mathbf{S}$ coincide on ω_T if the arbitrary data α, $\boldsymbol{\beta}$ coincide in the past of ω_T defined by $\mathbf{g}_1$, while the Cauchy data $(\boldsymbol{\phi}_1, \boldsymbol{\psi}_1)$ and $(\boldsymbol{\phi}_2, \boldsymbol{\psi}_2)$ coincide on ω_0, which is the intersection of the $\mathbf{g}_1$-past of ω_T with M_0.*

Proof. The difference of two solutions satisfies a linear homogeneous first order symmetric hyperbolic system with $\mathcal{C}^1$ principal coefficients and $\mathcal{C}^0$ non-principal ones. We set $\boldsymbol{\gamma} = \bar{\mathbf{\Gamma}}_1 - \bar{\mathbf{\Gamma}}_2$, $\mathbf{k} = \mathbf{K}_1 - \mathbf{K}_2$, $\mathbf{G} = \bar{\mathbf{g}}_1 - \bar{\mathbf{g}}_2$, $\mathbf{a} = \mathbf{A}_1 - \mathbf{A}_2$, and likewise for the Bianchi fields obtained from $\mathbf{A}$. The linear system is composed of three sets of equations in which we denote by ℓ various linear forms in $\boldsymbol{\gamma}$, $\mathbf{k}, \ldots$ with coefficients analytic in $\mathbf{u}_1$ and $\mathbf{u}_2$.

The first set is

$$\hat{\partial}_0 \gamma^h{}_{ij} + N_1 g_1^{hk} \partial_k k_{ij} = \ell^{(\gamma)}_{ij}, \tag{78}$$

$$\hat{\partial}_0 k_{ij} + N_1 \partial_h \gamma^h{}_{ij} = \ell^{(k)}_{ij}. \tag{79}$$

The second set consists of an equation for $\mathbf{e} = \mathbf{E}_1 - \mathbf{E}_2$

$$\hat{\partial}_0 e_{ij} - N_1 g_1^{hm} g_1^{ln} \eta_{1mni} h_{lj} = \ell^{(e)}_{ij} \tag{80}$$

and similar equations for $\mathbf{h} = \mathbf{H}_1 - \mathbf{H}_2$, $\mathbf{d} = \mathbf{D}_1 - \mathbf{D}_2$, and $\mathbf{b} = \mathbf{B}_1 - \mathbf{B}_2$ deduced from the Bianchi equations. [Note that $\mathbf{e}$ in (80) is not to be confused with the metric $\bar{\boldsymbol{e}}$ introduced in Sec. 8 and below in the discussion of $H_s^{\text{u.l.}}$.] The third set of equations is

$$\hat{\partial}_0 G_{ij} = \ell^{(G)}_{ij}. \tag{81}$$

Raising and lowering indices with the metric $\bar{\mathbf{g}}_1$, we multiply (78) by $\gamma_h{}^{ij}$, (79) by k^{ij}, (80) by e^{ij} and so on. We add the results and integrate over a domain Ω of the type indicated in the preceding Lemma, constructed with $\mathbf{g}_1$. We denote by $\mathcal{E}(\omega_t)$ the integral

$$\mathcal{E}(\omega_t) = \frac{1}{2} \int_{\omega_t} (|\boldsymbol{\gamma}|^2 + |\mathbf{k}|^2 + |\mathbf{e}|^2 + |\mathbf{h}|^2 + |\mathbf{d}|^2 + |\mathbf{b}|^2 + |\mathbf{G}|^2) \mu_{\bar{\mathbf{g}}_1}. \tag{82}$$

Integration of (82) over Ω leads to the identity

$$\mathcal{E}(\omega_T) = \mathcal{E}(\omega_0) + \int_L \boldsymbol{\sigma} + \int_0^T \int_{\omega_t} q \mu_{\bar{\mathbf{g}}_1(t)} dt, \tag{83}$$

where $\boldsymbol{\sigma}$ is a three-form and q is a quadratic function of $\boldsymbol{\gamma},\ \mathbf{k},\ldots,\mathbf{G}$ with coefficients analytic in $\mathbf{u}_1$ and $\mathbf{u}_2$. The three-form $\boldsymbol{\sigma}$ arises from integration of a $\hat{\partial}_0$ derivative together with a spatial divergence from the principal terms of our equations. We made an analogous computation for the Bianchi equations and showed that the integral over L in that case is positive. The same argument holds here for the terms coming from the second set of equations. It is easy to prove that the integral over L is also positive for the first and third sets of equations. On the other hand, q is bounded on Ω, up to a multiplicative constant C, by the sum of the squares of the $\bar{\mathbf{g}}_1$ norms of $\boldsymbol{\gamma},\ \mathbf{k},\ \mathbf{e},\ldots,\ \mathbf{G}$. Hence, we have the inequality

$$\mathcal{E}(\omega_T) \leq \mathcal{E}(\omega_0) + C\int_0^T \mathcal{E}(\omega_t)dt. \tag{84}$$

It follows that $\mathcal{E}(\omega_T) = 0$ if $\mathcal{E}(\omega_0) = 0$, which is the stated result.

By using Theorem 3, we will show that the local existence result holds for a non-compact manifold M without any hypotheses on the fall-off at infinity of the initial data.

Definition. *A tensor* $\mathbf{u}$ *on* M *belongs to a space* $H_s^{\mathrm{u.l.}}$ *("u.l." denotes "uniformly local") if there exists a covering* $\mathbf{C}$ *of* M *by open sets* $\omega_{(i)}$ *with compact closures such that*

1. *The restriction of* $\mathbf{u}$ *to each of these open sets belongs to* $H_s(\omega_{(i)})$*, that is,* $\mathbf{u}$ *is square-integrable on* $\omega_{(i)}$ *in the volume element defined by the metric* $\bar{\boldsymbol{e}}$ *introduced in Sec. 8, and so are all its* $\bar{\boldsymbol{e}}$*-covariant derivatives of order* $\leq s$*.*

2. $$\mathrm{Sup}_i \|\mathbf{u}\|_{H_s(\omega_{(i)})} \equiv \|\mathbf{u}\|_{H_s^{\mathrm{u.l.}}} < \infty.$$

Note that different choices of uniformly locally finite coverings, as well as different uniformly equivalent metrics $\bar{\boldsymbol{e}}$, define the same $H_s^{\mathrm{u.l.}}$ space because they define equivalent norms.

Lemma. *If, for the covering* $\mathbf{C}$*, the arbitrary quantities and the original initial data are such that*

$$\alpha \in \mathcal{C}^0(I, H_{s+2}^{\mathrm{u.l.}}), \quad \alpha > 0,$$

$$\boldsymbol{\beta} \in \mathcal{C}^0(I, H^{\text{u.l.}}_{s+1}),$$

$$\boldsymbol{\phi} \in H^{\text{u.l.}}_{s+2}, \quad \boldsymbol{\psi} \in H^{\text{u.l.}}_{s+1},$$

and $\boldsymbol{\phi}$ *defines a properly Riemannian metric on* M*, uniformly equivalent to* $\bar{\boldsymbol{e}}$*, then the initial data for* $\mathbf{u}$ *belong to* $H^{\text{u.l.}}_s$ *if* $s \geq 1$*.*

Proof. The proof follows from the definition and the Sobolev multiplication properties.

Theorem 4. *Hypotheses. a. The arbitrary quantities are such that, with* I *an interval containing* 0*,*

$$\alpha \in \mathcal{C}^0(I, H^{\text{u.l.}}_{s+2}), \quad \alpha > A > 0,$$

$$\boldsymbol{\beta} \in \mathcal{C}^0(I, H^{\text{u.l.}}_{s+1}).$$

b. The initial data are such that

$$\boldsymbol{\phi} \in H^{\text{u.l.}}_{s+2}, \quad \boldsymbol{\psi} \in H^{\text{u.l.}}_{s+1},$$

and $\boldsymbol{\phi}$ *defines a properly Riemannian metric on* M*, uniformly equivalent to* $\bar{\boldsymbol{e}}$*.*
Conclusion. If $s \geq 3$*, there exists a number* $T > 0$*,* $[0,T] \subset I$*, such that the system* $\mathbf{S}$ *has one and only one solution* $\mathbf{u} \in \mathcal{C}^0([0,T], H^{\text{u.l.}}_s) \cap \mathcal{C}^1([0,T], H^{\text{u.l.}}_{s-1})$ *taking the Cauchy data* $\mathbf{u}_0 \in H^{\text{u.l.}}_s$ *deduced from* $\boldsymbol{\phi}$*,* $\boldsymbol{\psi}$*.*

Proof. Suppose M is not compact. Consider a covering of M_T by compact sets $K^{(i)}_T$. The pasts of these sets in the globally hyperbolic metric $-A^2 dt^2 + B_2 \bar{\boldsymbol{e}}$ cover $V_T \equiv M \times [0,T]$. (The numbers A and B_2 here refer to Theorem 2.) The intersections of these pasts with M_0 define a covering of M_0 by compact sets $K^{(i)}_0$. There exist truncating functions $\theta^{(i)}$ that are smooth, bounded uniformly with respect to i (as are their $\bar{\boldsymbol{e}}$-covariant derivatives of order $\leq s$), equal to one on $K^{(i)}_0$ and equal to zero outside another compact set $\tilde{K}^{(i)}$ strictly containing $K^{(i)}_0$ [20]. Use the sets $\tilde{K}^{(i)}$ to define the $H^{\text{u.l.}}_s$ norms of tensors on M.

Consider the Cauchy problem for the system $\tilde{\mathbf{S}}$ with Cauchy data $\tilde{\mathbf{u}}^{(i)}_0 = \theta^{(i)} \tilde{\mathbf{u}}_0$. There exists a constant C independent of i such that

$$\|\tilde{\mathbf{u}}^{(i)}_0\|_{H_s} \leq C \|\tilde{\mathbf{u}}_0\|_{H^{\text{u.l.}}_s}.$$

Corresponding inequalities hold for the given quantities α, $\boldsymbol{\beta}$. Theorem 2 shows that there exists a number $T' < T$, independent of i, such that $\mathbf{S}$ has one and only one solution $\mathbf{u}^{(i)}$ on $M \times [0,T]$ that takes the initial value $\tilde{\mathbf{u}}_0^{(i)}$.

Remark. The associated metric $\bar{\mathbf{g}}^{(i)}$ is uniformly equivalent to $\bar{\boldsymbol{e}}$ and the spacetime metric $\mathbf{g}^{(i)}$ satisfies the inequalities posited in Theorem 2. Therefore, the past of a subset in this metric is contained in the past defined with $\bar{\boldsymbol{e}}$.

We obtain, from the set of $u^{(i)}$'s defined on V_T, a solution $\mathbf{u}$ for the Cauchy problem for $\mathbf{S}$ on V_t by setting

$$\mathbf{u}(x,t) = u^{(i)}(x,t) \text{ on } \mathcal{P}(K_T^{(i)}) \cap \{0 \leq t \leq T'\}$$

because, by Theorem 3 and the previous Remark, we have

$$u^{(j)}(x,t) = u^{(i)}(x,t) \text{ on } \mathcal{P}(K_T^{(j)}) \cap \mathcal{P}(K_T^{(i)}) \cap \{0 \leq t \leq T'\}.$$

Finally, we prove

Theorem 5. *The solution of* $\mathbf{S}$ *with the data deduced from the arbitrary quantities* α, $\boldsymbol{\beta}$ *and the initial data* $\boldsymbol{\phi}$, $\boldsymbol{\psi}$ *is a solution of the vacuum Einstein equations with lapse* $N = \alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$ *and shift* $\boldsymbol{\beta}$ *if the initial data satisfy the constraints.*

Proof. The result holds for H_{s+1} as well as for $H_{s+1}^{\text{u.l.}}$ data. We spell it out for the first case. It has been proven by Choquet-Bruhat and York[5], for initial data in the indicated function spaces and satisfying the constraints, and for a given arbitrary shift $\boldsymbol{\beta}$, that the vacuum Einstein equations have one and only one solution in the "algebraic gauge," that is, with a lapse function in the given form, such that $\bar{\mathbf{g}} \in \mathcal{C}^0(M \times [0,T])$, $(\mathbf{D}\bar{\mathbf{g}}, \mathbf{K}) \in \mathcal{C}^0([0,T], H_{s+1})$. The connection of $\bar{\mathbf{g}}$, the second fundamental form $\mathbf{K}$ and the Riemann tensor of $\bar{\mathbf{g}}$ satisfy the system $\mathbf{S}$ and take the Cauchy data $\mathbf{u}_0$. They coincide, therefore, with the solution $\mathbf{u}$ found previously. In particular, the component $\bar{\mathbf{g}}$ of $\mathbf{u}$ together with $N = \alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$ and $\boldsymbol{\beta}$ are a metric that satisfies the vacuum Einstein equations.

Remark. It is a result of the coincidence of the components $\bar{\mathbf{g}}$ and $\mathbf{K}$ of $\mathbf{u}$ with the solution of the Einstein equations with initial data $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$, given shift $\boldsymbol{\beta}$, and lapse $N = \alpha^{-1}(\det \bar{\mathbf{g}})^{1/2}$ that these components are in fact such that $(\mathbf{D}\bar{\mathbf{g}}, \mathbf{K}) \in \mathcal{C}^0([0,T], H_{s+1})$.

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